

AN OPTIMAL INEQUALITY ON LOCALLY STRONGLY CONVEX CENTROAFFINE HYPERSURFACES

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ABSTRACT. In this paper, we establish a general inequality for locally strongly convex centroaffine hypersurfaces in \mathbb{R}^{n+1} involving the norm of the covariant derivatives of both the difference tensor K and the Tchebychev vector field T . Our result is optimal in that, applying our recent classification for locally strongly convex centroaffine hypersurfaces with parallel cubic form in [4], we can completely classify the hypersurfaces which realize the equality case of the inequality.

1. INTRODUCTION

Let \mathbb{R}^{n+1} be the $(n+1)$ -dimensional affine space equipped with its canonical flat connection D and the parallel volume form \det . In this paper, we show that for locally strongly convex centroaffine hypersurfaces in \mathbb{R}^{n+1} there is an optimal inequality involving centroaffine invariants.

Recall that in centroaffine differential geometry, we study properties of hypersurfaces in \mathbb{R}^{n+1} that are invariant under the centroaffine transformation group G in \mathbb{R}^{n+1} . Here, by definition, G is the subgroup of affine transformation group in \mathbb{R}^{n+1} which keeps the origin $O \in \mathbb{R}^{n+1}$ invariant. Let M^n be an n -dimensional smooth manifold. An immersion $x : M^n \rightarrow \mathbb{R}^{n+1}$ is said to be a centroaffine hypersurface if, for each point $x \in M^n$, the position vector x (from O) is transversal to the tangent space $T_x M$ of M at x . In that situation, the position vector x defines the *centroaffine normalization* modulo orientation. For any vector fields X and Y tangent to M^n , we have the centroaffine formula of Gauss:

$$(1.1) \quad D_X x_*(Y) = x_*(\nabla_X Y) + h(X, Y)(-\varepsilon x),$$

where $\varepsilon = 1$ or -1 . Moreover, associated with (1.1) we will call $-\varepsilon x$, ∇ and h the centroaffine normal, the induced (centroaffine) connection and the centroaffine metric, respectively. In this paper, we will consider only locally strongly convex centroaffine hypersurfaces such that the bilinear 2-form h defined by (1.1) remains definite; then we will choose ε such that the centroaffine metric h is positive definite.

Let $x : M^n \rightarrow \mathbb{R}^{n+1}$ be a locally strongly convex centroaffine hypersurface and $\hat{\nabla}$ be the Levi-Civita connection of its centroaffine metric h . Then its difference tensor K is defined by $K(X, Y) := K_X Y := \nabla_X Y - \hat{\nabla}_X Y$; it is symmetric as both connections are torsion free. Define the cubic form C by $C := \nabla h$; it is related to

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the difference tensor by the equation

$$(1.2) \quad C(X, Y, Z) := (\nabla_X h)(Y, Z) = -2h(K_X Y, Z).$$

It follows that C is a totally symmetric tensor of type $(0, 3)$, and that $\hat{\nabla}K = 0$ is equivalent to $\hat{\nabla}C = 0$. Now, we define the Tchebychev form T^\sharp and its associated Tchebychev vector field T such that:

$$(1.3) \quad nT^\sharp(X) = \text{trace}(K_X), \quad h(T, X) = T^\sharp(X).$$

If $T = 0$, or equivalently, $\text{trace } K_X = 0$ for any tangent vector X , then M^n reduced to be the so-called *proper (equi-)affine hypersphere* centered at the origin O (cf. p.279 of [11], or see Section 1.15.2-3 therein for more details). Using the difference tensor K and the Tchebychev vector field T one can define a traceless difference tensor \tilde{K} by

$$(1.4) \quad \tilde{K}(X, Y) := K(X, Y) - \frac{n}{n+2}[h(X, Y)T + h(X, T)Y + h(Y, T)X].$$

It is well-known that \tilde{K} vanishes if and only if $x(M^n)$ lies in a hyperquadric (cf. Section 7.1 in [18]; Lemma 2.1 and Remark 2.2 in [9]; refer also to [1] and its reviewer's comments in MR2155181).

Now, we can state the main result of this paper as follows:

Theorem 1.1. *Let $x : M^n \rightarrow \mathbb{R}^{n+1}$ be a locally strongly convex centroaffine hypersurface. Then the difference tensor K and the Tchebychev vector field T of M^n satisfy the following inequality*

$$(1.5) \quad \|\hat{\nabla}K\|^2 \geq \frac{3n^2}{n+2}\|\hat{\nabla}T\|^2,$$

where $\|\cdot\|$ denotes the tensorial norm with respect to the centroaffine metric h . Moreover, the equality holds at every point of M^n if and only if $\hat{\nabla}\tilde{K} = 0$, and one of the following cases occurs:

- (i) $x(M^n)$ is an open part of a locally strongly convex hyperquadric; or
- (ii) $x(M^n)$ is obtained as the (generalized) Calabi product of a lower dimensional locally strongly convex centroaffine hypersurface with parallel cubic form and a point; or
- (iii) $x(M^n)$ is obtained as the (generalized) Calabi product of two lower dimensional locally strongly convex centroaffine hypersurfaces with parallel cubic form; or
- (iv) $n = \frac{1}{2}m(m+1) - 1$, $m \geq 3$, $x(M^n)$ is centroaffinely equivalent to the standard embedding of $\text{SL}(m, \mathbb{R})/\text{SO}(m) \hookrightarrow \mathbb{R}^{n+1}$; or
- (v) $n = \frac{1}{4}(m+1)^2 - 1$, $m \geq 5$, $x(M^n)$ is centroaffinely equivalent to the standard embedding $\text{SL}(\frac{m+1}{2}, \mathbb{C})/\text{SU}(\frac{m+1}{2}) \hookrightarrow \mathbb{R}^{n+1}$; or
- (vi) $n = \frac{1}{8}(m+1)(m+3) - 1$, $m \geq 9$, $x(M^n)$ is centroaffinely equivalent to the standard embedding $\text{SU}^*(\frac{m+3}{2})/\text{Sp}(\frac{m+3}{4}) \hookrightarrow \mathbb{R}^{n+1}$; or
- (vii) $n = 26$, $x(M^n)$ is centroaffinely equivalent to the standard embedding $\text{E}_{6(-26)}/\text{F}_4 \hookrightarrow \mathbb{R}^{27}$; or
- (viii) $x(M^n)$ is locally centroaffinely equivalent to the canonical centroaffine hypersurface $x_{n+1} = \frac{1}{2x_1} \sum_{k=2}^n x_k^2 + x_1 \ln x_1$.

Remark 1.1. For detailed discussions about all the above examples, namely the notion of (*generalized*) *Calabi product* and the standard embedding, the readers are referred to [4] (cf. also [7]). We should point it out that the ellipsoids and the hyperboloids which are centered at the origin O , and also the hypersurfaces in (ii)-(viii), have parallel cubic form, i.e., $\hat{\nabla}C = 0$ or equivalently $\hat{\nabla}K = 0$; while a hyperquadric with no center or not being centered at the origin O has the properties that $K \neq 0$ and $\hat{\nabla}K \neq 0$ (cf. [4]). We also remark that a centroaffine hypersurface is called *canonical* meaning that its centroaffine metric h is flat and its cubic form C satisfies $\hat{\nabla}C = 0$ (cf. [13]).

Remark 1.2. The lists of centroaffine hypersurfaces as shown in Theorem 1.1 give the classification of centroaffine hypersurfaces in \mathbb{R}^{n+1} with parallel traceless cubic form (which is equivalent to $\hat{\nabla}\tilde{K} = 0$) for every $n \geq 2$. This is a complete extension of [15] where the classification was achieved only for $n = 2$. On the other hand, locally strongly convex centroaffine hypersurfaces with $\hat{\nabla}K = 0$ are classified in [4] for every dimensions.

Remark 1.3. Besides that as stated in [4], different characterizations on the typical examples of centroaffine hypersurfaces appearing in Theorem 1.1 were established in our recent articles, [2] and [3], from other aspects of differential geometric invariants.

Remark 1.4. Related with the study of centroaffine hypersurfaces, with pleasure we would like to introduce the interesting results of Li, Simon and Zhao [10] and also the very recent development due to Cortés, Nardmann and Suhr [5], where among other important results the authors investigated the problem under what conditions a locally strongly convex centroaffine hypersurface is complete with respect to the centroaffine metric.

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2. PRELIMINARIES

In this section, we briefly recall some basic facts about centroaffine hypersurfaces. We refer to [8], [11, 17], [18] and [13, 20] for more detailed discussions.

Given a centroaffine hypersurface M^n , we choose an h -orthonormal tangential frame field $\{e_1, \dots, e_n\}$. Let $\{\theta_1, \dots, \theta_n\}$ be its dual frame field and $\{\theta_{ij}\}$ its Levi-Civita connection forms. Let K_{ij}^k and T^i denote the components of K and T with respect to $\{e_i\}$. Then (1.4) can be written as

$$(2.1) \quad \tilde{K}_{ij}^k := K_{ij}^k - \frac{n}{n+2}(T^k \delta_{ij} + T^i \delta_{jk} + T^j \delta_{ik}),$$

where $K_{ij}^k = h(K_{e_i} e_j, e_k)$, $T^i = \frac{1}{n} \sum_j K_{jj}^i$.

Let $K_{ij,l}^k$ and T_i^j be the components of the covariant differentiation $\hat{\nabla}K$ and $\hat{\nabla}T$, respectively, which by definition can be expressed by

$$\sum_l K_{ij,l}^k \theta_l = dK_{ij}^k + \sum_l K_{lj}^k \theta_{li} + \sum_l K_{il}^k \theta_{lj} + \sum_l K_{ij}^l \theta_{lk},$$

$$\sum_i T_{,i}^j \theta_i = dT^j + \sum_i T^i \theta_{ij}.$$

Denote by \hat{R}_{ijkl} the components of the Riemannian curvature tensor of the centroaffine metric h . Then, we have the equations of Gauss and Codazzi as follows:

$$(2.2) \quad \hat{R}_{ijkl} = \varepsilon(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_m (K_{il}^m K_{jk}^m - K_{ik}^m K_{jl}^m),$$

$$(2.3) \quad K_{ij,l}^k = K_{ij,k}^l, \quad 1 \leq i, j, k, l \leq n.$$

3. THE INEQUALITY AND SOME RELATED LEMMAS

We start with the following result.

Proposition 3.1. *Let $x : M^n \rightarrow \mathbb{R}^{n+1}$ be a locally strongly convex centroaffine hypersurface. Then*

$$(3.1) \quad \|\hat{\nabla} K\|^2 \geq \frac{3n^2}{n+2} \|\hat{\nabla} T\|^2,$$

where $\|\hat{\nabla} K\|^2 = \sum (K_{ij,l}^k)^2$, $\|\hat{\nabla} T\|^2 = \sum (T_{,i}^j)^2$. Moreover, the equality holds in (3.1) if and only if the traceless difference tensor \tilde{K} is parallel, i.e., $\hat{\nabla} \tilde{K} = 0$, or equivalently:

$$(3.2) \quad K_{ij,l}^k = \frac{n}{n+2} (T_{,l}^k \delta_{ij} + T_{,l}^i \delta_{jk} + T_{,l}^j \delta_{ik}), \quad 1 \leq i, j, k, l \leq n.$$

Proof. From the definition (1.4) or (2.1), we have

$$(3.3) \quad \tilde{K}_{ij,l}^k = K_{ij,l}^k - \frac{n}{n+2} (T_{,l}^k \delta_{ij} + T_{,l}^i \delta_{jk} + T_{,l}^j \delta_{ik}).$$

It is easy to check that

$$(3.4) \quad \begin{aligned} 0 \leq \|\hat{\nabla} \tilde{K}\|^2 &:= \sum (\tilde{K}_{ij,l}^k)^2 = \sum (K_{ij,l}^k)^2 - \frac{3n^2}{n+2} \sum (T_{,i}^j)^2 \\ &= \|\hat{\nabla} K\|^2 - \frac{3n^2}{n+2} \|\hat{\nabla} T\|^2. \end{aligned}$$

Obviously, equality in (3.1) holds if and only if $\|\hat{\nabla} \tilde{K}\| = 0$, i.e., it holds $\tilde{K}_{ij,l}^k = 0$, $1 \leq i, j, k, l \leq n$, which is equivalent to (3.2). \square

Next, we investigate the implications if (3.2) holds.

Lemma 3.1. *Let $x : M^n \rightarrow \mathbb{R}^{n+1}$ be a locally strongly convex centroaffine hypersurface. If (3.2) holds, then we have $T_{,k}^j = \frac{1}{n} \sum T_{,i}^i \delta_{jk}$, $1 \leq j, k \leq n$, namely,*

$$(3.5) \quad \hat{\nabla} T = \lambda \cdot \text{id}, \quad \lambda = \frac{1}{n} \sum T_{,i}^i.$$

Proof. Exchanging k with l in (3.2), we have

$$(3.6) \quad K_{ij,k}^l = \frac{n}{n+2} (T_{,k}^l \delta_{ij} + T_{,k}^i \delta_{jl} + T_{,k}^j \delta_{il}).$$

Combining (2.3), (3.2) and (3.6), we obtain

$$(3.7) \quad T_{,k}^l \delta_{ij} + T_{,k}^i \delta_{jl} + T_{,k}^j \delta_{il} = T_{,l}^k \delta_{ij} + T_{,l}^i \delta_{jk} + T_{,l}^j \delta_{ik}, \quad 1 \leq i, j, k, l \leq n.$$

Taking the summation for $i = l$ in (3.7) and noting that $T_{,k}^j = T_{,j}^k$, we get

$$(3.8) \quad T_{,k}^j = \frac{1}{n} \sum T_{,i}^i \delta_{jk}, \quad 1 \leq j, k \leq n.$$

This verifies the assertion. \square

Remark 3.1. If (3.5) holds, then T is a conformal vector field and M^n by definition is called a Tchebychev hypersurface (cf. [14]). Therefore, if the equality holds in (3.1) then M^n is a Tchebychev hypersurface.

Lemma 3.2. *Let $x : M^n \rightarrow \mathbb{R}^{n+1}$ be a locally strongly convex centroaffine hypersurface. Then, (3.2) holds if and only if it holds that*

$$(3.9) \quad K_{ij,l}^k = \mu(\delta_{kl}\delta_{ij} + \delta_{il}\delta_{jk} + \delta_{jl}\delta_{ik}), \quad 1 \leq i, j, k, l \leq n,$$

where $\mu = \frac{1}{n+2} \sum T_{,l}^l$.

Proof. For the “if” part, we assume that (3.9) holds. By summing over $i = j$ in (3.9), we get

$$(3.10) \quad T_{,l}^k = \frac{n+2}{n} \mu \delta_{kl}, \quad 1 \leq k, l \leq n.$$

Then (3.2) immediately follows.

Conversely, for the “only if” part, we assume that (3.2) holds. Then we have (3.8), and therefore (3.9) holds with $\mu = \frac{1}{n+2} \sum_l T_{,l}^l$. \square

Now, we fix a point $p \in M^n$. For subsequent purpose, we will review the well-known construction of a typical orthonormal basis with respect to the centroaffine metric h for $T_p M^n$, which was introduced by Ejiri and has been widely applied, and proved to be very useful for various situations, see e.g. [6] and [12, 16]. The idea is to construct from the $(1, 2)$ tensor K a self adjoint operator at a point; then one extends the eigenbasis to a local field.

Let $p \in M^n$ and $U_p M^n = \{u \in T_p M^n \mid h(u, u) = 1\}$. Since M^n is locally strongly convex, $U_p M^n$ is compact. We define a function f on $U_p M^n$ by $f(u) = h(K_u u, u)$. Then there is an element $e_1 \in U_p M^n$ at which the function $f(u)$ attains an absolute maximum, denoted by λ_1 . Then we have the following lemma. For its proof, we refer the reader to [6].

Lemma 3.3 ([6]). *There exists an orthonormal basis $\{e_1, \dots, e_n\}$ of $T_p M^n$ such that the following hold:*

- (i) $K_{e_1} e_i = \lambda_i e_i$, for $i = 1, \dots, n$.
- (ii) $\lambda_1 \geq 2\lambda_i$, for $i \geq 2$. If $\lambda_1 = 2\lambda_i$, then $f(e_i) = 0$.

When working at the point $p \in M^n$, we will always assume that an orthonormal basis is chosen such that Lemma 3.3 is satisfied. While if we work at a neighborhood of $p \in M^n$ and if not stated otherwise, we will choose an h -orthonormal frame field $\{E_1, \dots, E_n\}$ such that $E_1(p) = e_1, \dots, E_n(p) = e_n$, and $\{e_1, \dots, e_n\}$ is chosen as in Lemma 3.3.

The following lemma is crucial for our proof of Theorem 1.1.

Lemma 3.4. *Let $x : M^n \rightarrow \mathbb{R}^{n+1}$ be a locally strongly convex centroaffine hypersurface. If (3.9) holds, then we have*

$$(3.11) \quad e_l(\mu) = 0, \quad 2 \leq l \leq n,$$

$$(3.12) \quad e_1(\mu) = (2\lambda_l - \lambda_1)(\lambda_l^2 - \lambda_1 \lambda_l + \varepsilon), \quad 2 \leq l \leq n,$$

$$(3.13) \quad (2\lambda_k - \lambda_1)(\lambda_l - \lambda_j)K_{jk}^l = 0, \quad 2 \leq j \neq l \leq n, \quad 1 \leq k \leq n,$$

$$(3.14) \quad (\lambda_l^2 - \lambda_1 \lambda_l + \varepsilon)K_{ll}^l = 0, \quad 2 \leq l \leq n.$$

Proof. Taking the covariant derivative of (3.9) implies that

$$(3.15) \quad K_{ij,lq}^k = e_q(\mu)(\delta_{kl}\delta_{ij} + \delta_{il}\delta_{jk} + \delta_{jl}\delta_{ik}).$$

Exchanging l with q in (3.15), we have

$$(3.16) \quad K_{ij,q l}^k = e_l(\mu)(\delta_{kq}\delta_{ij} + \delta_{iq}\delta_{jk} + \delta_{jq}\delta_{ik}).$$

From (3.15), (3.16) and the Ricci identity, we use (2.2) to obtain

$$(3.17) \quad \begin{aligned} & e_q(\mu)(\delta_{kl}\delta_{ij} + \delta_{il}\delta_{jk} + \delta_{jl}\delta_{ik}) - e_l(\mu)(\delta_{kq}\delta_{ij} + \delta_{iq}\delta_{jk} + \delta_{jq}\delta_{ik}) \\ &= \sum_m K_{mj}^k [\varepsilon(\delta_{iq}\delta_{lm} - \delta_{il}\delta_{qm}) + \sum_r (K_{il}^r K_{qm}^r - K_{qi}^r K_{lm}^r)] \\ &+ \sum_m K_{mi}^k [\varepsilon(\delta_{jq}\delta_{lm} - \delta_{jl}\delta_{qm}) + \sum_r (K_{jl}^r K_{qm}^r - K_{qj}^r K_{lm}^r)] \\ &- \sum_m K_{ij}^m [\varepsilon(\delta_{mq}\delta_{kl} - \delta_{ml}\delta_{kq}) + \sum_r (K_{ml}^r K_{kq}^r - K_{mq}^r K_{lk}^r)]. \end{aligned}$$

Taking $i = j = q = 1$ and $l \geq 2$ in (3.17), we obtain that

$$(3.18) \quad e_1(\mu)\delta_{kl} - 3e_l(\mu)\delta_{k1} = (2\lambda_k - \lambda_1)(\lambda_k^2 - \lambda_1\lambda_k + \varepsilon)\delta_{kl}.$$

First, letting $k = 1$ in (3.18), we get

$$e_l(\mu) = 0, \quad 2 \leq l \leq n.$$

Next, letting $k = l \geq 2$ in (3.18), we have

$$e_1(\mu) = (2\lambda_l - \lambda_1)(\lambda_l^2 - \lambda_1\lambda_l + \varepsilon), \quad 2 \leq l \leq n.$$

Then, letting $i = j = 1$ and $2 \leq q \neq l \leq n$ in (3.17), combining with (3.11), we obtain

$$(2\lambda_k - \lambda_1)(\lambda_l - \lambda_q)K_{qk}^l = 0, \quad 2 \leq q \neq l \leq n, \quad 1 \leq k \leq n.$$

Finally, letting $i = q = 1$ and $j = k = l \geq 2$ in (3.17), a direct calculation gives

$$(\lambda_l^2 - \lambda_1\lambda_l + \varepsilon)K_{ll}^l = 0, \quad 2 \leq l \leq n.$$

We have completed the proof of Lemma 3.4. \square

4. PROOF OF THE MAIN THEOREM

In this section, we will complete the proof of Theorem 1.1. Let $x : M^n \rightarrow \mathbb{R}^{n+1}$ be a locally strongly convex centroaffine hypersurface. Then, according to Proposition 3.1 and Lemma 3.2, to prove Theorem 1.1 we are left to consider the case that (3.9) holds identically for some function μ on M^n .

4.1. (3.9) holds with $\mu \neq \text{constant}$. In this subsection, we consider n -dimensional locally strongly convex centroaffine hypersurfaces such that (3.9) holds identically with $\mu \neq \text{constant}$. Since our result is local in nature, the non-constancy of μ allows us to assume that $U_1 := \{q \in M^n \mid X(\mu) = 0, \forall X \in T_q M^n\}$ is not an open subset. Therefore, from now on we will carry our discussion in the following open dense subset of M^n :

$$M' = \{q \in M^n \mid \text{there exists } X \in T_q M^n \text{ such that } X(\mu) \neq 0\}.$$

First of all, we have the following lemma.

Lemma 4.1. *If (3.9) holds at every point of M^n with $\mu \neq \text{constant}$, then with respect to the orthonormal basis as stated in Lemma 3.3, the number of the distinct eigenvalues of K_{e_1} can be at most 3, so that it equals 2 or 3.*

Proof. Let $p \in M'$. From Lemma 3.4, we have

$$(4.1) \quad e_1(\mu) = (2\lambda_l - \lambda_1)(\lambda_l^2 - \lambda_1\lambda_l + \varepsilon) \neq 0, \quad 2 \leq l \leq n.$$

It follows that

$$(4.2) \quad \lambda_1 > 0, \quad \lambda_1 - 2\lambda_l > 0, \quad \lambda_l^2 - \lambda_1\lambda_l + \varepsilon \neq 0, \quad 2 \leq l \leq n,$$

and, for each $2 \leq l \leq n$, $y_l = \lambda_1 - 2\lambda_l$ satisfies the following equation in y :

$$(4.3) \quad e_1(\mu) + \frac{1}{4}y(y^2 + 4\varepsilon - \lambda_1^2) = 0, \quad y > 0.$$

Now, about the solution y of (4.3), we consider the following three cases:

- (1) If $4\varepsilon - \lambda_1^2 \geq 0$, then (4.3) shows that $e_1(\mu) < 0$. In this case, as an equation of y , (4.3) has only one positive solution. This implies that we have $\lambda_2 = \dots = \lambda_n$.
- (2) If $4\varepsilon - \lambda_1^2 < 0$ and $e_1(\mu) < 0$, then again (4.3) has only one positive solution y and that $\lambda_2 = \dots = \lambda_n$.
- (3) If $4\varepsilon - \lambda_1^2 < 0$ and $e_1(\mu) > 0$, then (4.3) has at most two positive solutions. This implies that at most two of $\{\lambda_2, \dots, \lambda_n\}$ are distinct.

On the other hand, from (4.2) we easily see that $\lambda_1 > \lambda_l$ for all $l \geq 2$.

This clearly completes the proof of Lemma 4.1. \square

As a direct consequence of Lemma 4.1, the study of centroaffine hypersurfaces such that (3.9) holds identically with $\mu \neq \text{constant}$ can be divided into two cases:

Case (i). $\lambda_2 = \dots = \lambda_m < \lambda_{m+1} = \dots = \lambda_n$, $2 \leq m \leq n-1$.

Case (ii). $\lambda_2 = \dots = \lambda_n$.

The following lemma is important in sequel of this subsection.

Lemma 4.2. *If (3.9) holds at every point of M^n with $\mu \neq \text{constant}$, then, for $\{e_i\}$ as described in Lemma 3.3, the difference tensor K takes the following form:*

$$(4.4) \quad K_{e_1}e_1 = \lambda_1e_1, \quad K_{e_i}e_i = \lambda_ie_i, \quad K_{e_i}e_j = \lambda_i\delta_{ij}e_1, \quad i, j = 2, \dots, n,$$

Proof. We separate the proof into two cases as above.

If **Case (i)** occurs, then from (3.14) and (4.2), we get

$$h(K_{e_l}e_l, e_l) = 0, \quad 2 \leq l \leq m.$$

It follows that

$$(4.5) \quad h(K_{e_i}e_j, e_k) = 0, \quad 2 \leq i, j, k \leq m.$$

On the other hand, from (3.13) and (4.2), we obtain

$$(4.6) \quad h(K_{e_i}e_j, e_k) = 0, \quad 2 \leq i, j \leq m, \quad m+1 \leq k \leq n.$$

Combining (4.5), (4.6) and the fact $h(K_{e_i}e_j, e_1) = \lambda_i\delta_{ij}$, we get the assertion

$$(4.7) \quad K_{e_i}e_j = \lambda_i\delta_{ij}e_1, \quad 2 \leq i, j \leq m.$$

Similarly, we can prove that

$$(4.8) \quad K_{e_i}e_j = \lambda_i\delta_{ij}e_1, \quad m+1 \leq i, j \leq n.$$

From (3.13) and (4.2) again, we have

$$(4.9) \quad h(K_{e_i}e_j, e_k) = 0, \quad 2 \leq i \leq m, \quad m+1 \leq j \leq n, \quad 1 \leq k \leq n.$$

This shows that

$$K_{e_i}e_j = 0, \quad 2 \leq i \leq m, \quad m+1 \leq j \leq n.$$

In summary, we have completed the proof of Lemma 4.2 for **Case (i)**.

Next, similar to the proof of (4.5), we can verify the assertion for **Case (ii)**. \square

To treat the above two cases separately, we first state the following result.

Lemma 4.3. *Case (i) does not occur.*

Proof. Suppose on the contrary that **Case (i)** does occur. Then, from (3.11) and (4.1), we get

$$(4.10) \quad h(\text{grad } \mu, e_1) = e_1(\mu) \neq 0, \quad h(\text{grad } \mu, e_l) = e_l(\mu) = 0, \quad 2 \leq l \leq n.$$

It follows that $e_1 = \pm \frac{\text{grad } \mu}{\|\text{grad } \mu\|}(p)$. Without loss of generality, we assume that $e_1 = \frac{\text{grad } \mu}{\|\text{grad } \mu\|}(p)$.

Now, in a neighborhood U around p , we define a unit vector field $E_1 = \frac{\text{grad } \mu}{\|\text{grad } \mu\|}$. It is easily seen from the proof of (4.10) that, for each $q \in U$, the function f should achieve its absolute maximum over $U_q M^n$ exactly at $E_1(q)$. Furthermore, the continuity of eigenvalue functions of K_{E_1} (cf. [19]) and Lemma 4.1 imply that the multiplicity of each of its eigenvalue functions is constant. Then applying Lemma 1.2 of [19] we have a smooth eigenvector extension of K_{E_1} , from $\{e_1, e_2, \dots, e_n\}$ at p to $\{E_1(q), E_2(q), \dots, E_n(q)\}$ at any point q in a neighborhood of p , such that $K_{E_1}E_i = \tilde{\lambda}_i E_i$, with the functions $\{\tilde{\lambda}_i\}_{i=1}^n$ satisfying $\tilde{\lambda}_1 \geq 2\tilde{\lambda}_i$ for $i \geq 2$ and

$$\tilde{\lambda}_2 = \dots = \tilde{\lambda}_m < \tilde{\lambda}_{m+1} = \dots = \tilde{\lambda}_n, \quad 2 \leq m \leq n-1.$$

It is easy to see that, with respect to the local h -orthonormal frame field $\{E_i\}_{i=1}^n$ and the eigenvalue functions $\{\tilde{\lambda}_i\}_{i=1}^n$, the foregoing lemmas that from Lemma 3.3 up to Lemma 4.2 remain valid.

Now, applying Lemma 4.2, we obtain

$$(4.11) \quad \begin{aligned} (\hat{\nabla}_{E_i} K)(E_1, E_1) &= \hat{\nabla}_{E_i} K(E_1, E_1) - 2K(\hat{\nabla}_{E_i} E_1, E_1) \\ &= \hat{\nabla}_{E_i} \tilde{\lambda}_1 E_1 - 2 \sum_{k=2}^n \tilde{\lambda}_k h(\hat{\nabla}_{E_i} E_1, E_k) E_k \\ &= E_i(\tilde{\lambda}_1) E_1 + \sum_{k=2}^n (\tilde{\lambda}_1 - 2\tilde{\lambda}_k) h(\hat{\nabla}_{E_i} E_1, E_k) E_k, \quad i \geq 2, \end{aligned}$$

and

$$(4.12) \quad \begin{aligned} (\hat{\nabla}_{E_i} K)(E_i, E_i) &= \hat{\nabla}_{E_i} K(E_i, E_i) - 2K(\hat{\nabla}_{E_i} E_i, E_i) \\ &= \hat{\nabla}_{E_i} \tilde{\lambda}_i E_1 - 2\tilde{\lambda}_i h(\hat{\nabla}_{E_i} E_i, E_1) E_i - 2\tilde{\lambda}_i h(\hat{\nabla}_{E_i} E_i, E_i) E_1 \\ &= E_i(\tilde{\lambda}_i) E_1 + 3\tilde{\lambda}_i h(\hat{\nabla}_{E_i} E_1, E_i) E_i \\ &\quad + \sum_{k \neq i} \tilde{\lambda}_i h(\hat{\nabla}_{E_i} E_1, E_k) E_k, \quad i \geq 2. \end{aligned}$$

Then, from (3.9), (4.11), (4.12) and the definition of $K_{ij,l}^k$, we obtain that

$$(4.13) \quad \mu = K_{11,i}^i = (\tilde{\lambda}_1 - 2\tilde{\lambda}_i) h(\hat{\nabla}_{E_i} E_1, E_i), \quad 2 \leq i \leq n,$$

$$(4.14) \quad 3\mu = K_{ii,i}^i = 3\tilde{\lambda}_i h(\hat{\nabla}_{E_i} E_1, E_i), \quad 2 \leq i \leq n.$$

From (4.13), (4.14), and noting that $\tilde{\lambda}_1 - 2\tilde{\lambda}_i \neq 0$ for $2 \leq i \leq n$, we finally get

$$(4.15) \quad \tilde{\lambda}_1 = 3\tilde{\lambda}_i, \quad 2 \leq i \leq n.$$

Hence, we have $\tilde{\lambda}_2 = \dots = \tilde{\lambda}_n$. This is a contradiction to **Case (i)**. \square

According to Lemmas 4.1 and 4.3, we see that if (3.9) holds at every point of M^n with $\mu \neq \text{constant}$, then **Case (ii)** should occur at every point of M' . Moreover, we can prove the following lemma.

Lemma 4.4. *If (3.9) holds at every point of M^n with $\mu \neq \text{constant}$, then there exists a local h -orthonormal frame field $\{E_1, \dots, E_n\}$ and a smooth non-vanishing function λ such that the difference tensor K takes the following form:*

$$(4.16) \quad K_{E_1} E_1 = 3\lambda E_1, \quad K_{E_1} E_i = \lambda E_i, \quad K_{E_i} E_j = \lambda \delta_{ij} E_1, \quad i, j = 2, \dots, n.$$

Proof. First of all, we see that (4.10) still holds, and without loss of generality we may assume that $e_1 = \frac{\text{grad } \mu}{\|\text{grad } \mu\|}(p)$. Now we define $E_1 = \frac{\text{grad } \mu}{\|\text{grad } \mu\|}$. Similar as in the proof of Lemma 4.3, for each point q in a neighborhood U of p , the function f should achieve its absolute maximum over $U_q M^n$ exactly at $E_1(q)$. Moreover, due to that $K_{E_1}(q)$ has exactly two distinct eigenvalues with multiplicities 1 and $n-1$, respectively, we can apply Lemma 1.2 of [19] again to obtain local orthonormal eigenvector fields of K_{E_1} , extending from $\{e_1, e_2, \dots, e_n\}$ at p to $\{E_1, E_2, \dots, E_n\}$ around p , such that $K_{E_1} E_i = \tilde{\lambda}_i E_i$, with the eigenvalue functions $\{\tilde{\lambda}_i\}_{i=1}^n$ satisfy $\tilde{\lambda}_2 = \dots = \tilde{\lambda}_n$.

It is easily seen that, with respect to $\{E_i\}_{i=1}^n$ and $\{\tilde{\lambda}_i\}_{i=1}^n$, the foregoing lemmas, from Lemma 3.3 up to Lemma 4.2, and that the equations from (4.11) up to (4.15), are still valid. Hence, we have $\tilde{\lambda}_1 = 3\tilde{\lambda}_i$ for $i \geq 2$.

This completes the proof of Lemma 4.4. \square

4.2. (3.9) holds with $\mu = \text{constant}$. In this subsection, we consider n -dimensional locally strongly convex centroaffine hypersurfaces such that (3.9) holds identically with $\mu = \text{constant}$. The following Proposition is the main result of this subsection.

Proposition 4.1. *Let $x : M^n \rightarrow \mathbb{R}^{n+1}$ be a locally strongly convex centroaffine hypersurface. If (3.9) holds at every point of M^n with $\mu = \text{constant}$, then $\mu = 0$ and M^n is of parallel cubic form.*

Proof. We first fix a point $p \in M^n$, and then we choose an orthonormal basis $\{e_i\}_{i=1}^n$ as in Lemma 3.3 such that

$$(4.17) \quad K_{e_1} e_i = \lambda_i e_i, \quad i = 1, \dots, n.$$

We take a geodesic $\gamma(s)$ passing through p in the direction of e_1 . Let $\{E_1, \dots, E_n\}$ be parallel vector fields along γ , such that $E_i(p) = e_i$, $1 \leq i \leq n$, and $E_1 = \gamma'(s)$. Then we have $h(E_i, E_j) = h(e_i, e_j) = \delta_{ij}$ for $1 \leq i, j \leq n$.

Applying (3.9), we get that

$$(4.18) \quad \frac{\partial}{\partial s} h(K(E_1, E_1), E_i) = h((\hat{\nabla}_{E_1} K)(E_1, E_1), E_i) = 0, \quad 2 \leq i \leq n,$$

$$(4.19) \quad \frac{\partial}{\partial s} h(K(E_1, E_i), E_j) = h((\hat{\nabla}_{E_1} K)(E_1, E_i), E_j) = 0, \quad 2 \leq i \neq j \leq n.$$

Then we have

$$(4.20) \quad \begin{cases} h(K(E_1, E_1), E_i) = h(K(e_1, e_1), e_i) = 0, \\ h(K(E_1, E_i), E_j) = h(K(e_1, e_i), e_j) = 0, \end{cases} \quad 2 \leq i \neq j \leq n.$$

It follows that there exist functions $\tilde{\lambda}_i$ ($1 \leq i \leq n$) defined along γ , such that

$$(4.21) \quad K_{E_1} E_i = \tilde{\lambda}_i E_i, \quad \tilde{\lambda}_i(p) = \lambda_i, \quad i = 1, \dots, n.$$

Now, due to (4.21) and that $\mu = \text{constant}$, we can follow the proof of (3.12) to show that, along γ ,

$$(4.22) \quad (2\tilde{\lambda}_i - \tilde{\lambda}_1)(\tilde{\lambda}_i^2 - \tilde{\lambda}_1 \tilde{\lambda}_i + \varepsilon) = 0, \quad 2 \leq i \leq n.$$

Applying (3.9) again, we get

$$(4.23) \quad \begin{cases} \frac{\partial}{\partial s} \tilde{\lambda}_1 = \frac{\partial}{\partial s} h(K(E_1, E_1), E_1) = h((\hat{\nabla}_{E_1} K)(E_1, E_1), E_1) = 3\mu, \\ \frac{\partial}{\partial s} \tilde{\lambda}_i = \frac{\partial}{\partial s} h(K(E_1, E_i), E_i) = h((\hat{\nabla}_{E_1} K)(E_1, E_i), E_i) = \mu, \quad 2 \leq i \leq n. \end{cases}$$

By (4.23), taking the derivative of (4.22) three times along $\gamma(s)$ implies that

$$(4.24) \quad 12\mu^3 = 0.$$

This combining with (3.9) clearly implies that M^n has parallel cubic form. \square

4.3. Completion of the proof of Theorem 1.1. As we have already stated in the beginning of Section 4, to prove Theorem 1.1, we are left to consider the case that (3.9) holds identically for some function μ on M^n . Now, we should consider two cases: $\mu \neq \text{constant}$, or $\mu = \text{constant}$.

(1) If $\mu \neq \text{constant}$, then we can apply Lemma 4.4 to obtain that

$$T^1 = \frac{n+2}{n} \lambda \neq 0, \quad T^2 = \dots = T^n = 0,$$

which, by (4.16), further implies that $K_{ij}^k = \frac{n}{n+2}(T^k \delta_{ij} + T^i \delta_{jk} + T^j \delta_{ik})$. This implies that $\tilde{K} = 0$. According to subsection 7.1.1 of [18], and also Lemma 2.1 of [9] and noting that $K \neq 0$, we easily see that locally M^n is a hyperquadric, which either has no center, or is not centered at the origin.

(2) If $\mu = \text{constant}$, then by Proposition 4.1, M^n is of parallel cubic form. It follows that we can apply the (classification) Theorem 1.1 of [4] to see that locally M^n is either a hyperquadric with the origin as its center (i.e. $K = 0$), or one of the hypersurfaces as stated from (ii) up to (viii) of Theorem 1.1.

We have completed the proof of Theorem 1.1. \square

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